# Exponential Smoothing as State Space

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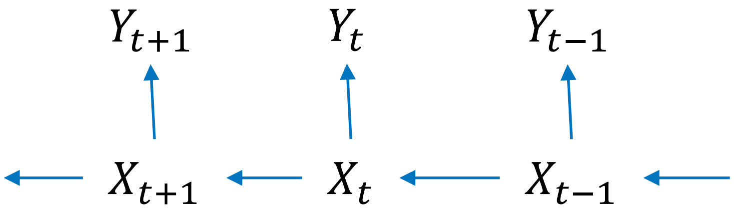
# Exponential Smoothing Models

- Simple Exponential Smoothing
  - $\mathcal{F}_{t+1} = \alpha Y_t + (1-\alpha)\mathcal{F}_t$
- Holt's Linear
  - $\mathcal{F}_{t+1} = \alpha Y_t + (1-\alpha)(\mathcal{F}_t + T_t)$
  - $T_{t+1} = \gamma(\mathcal{F}_{t+1} - \mathcal{F}_t) + (1-\gamma)T_t$
- Holt-Winters (additive)
  - $\mathcal{F}_{t+1} = \alpha(Y_t - S_{t-m}) + (1-\alpha)(\mathcal{F}_t + T_t)$
  - $T_{t+1} = \gamma(\mathcal{F}_{t+1} - \mathcal{F}_t) + (1-\gamma)T_t$
  - $S_{t+1} = \delta(Y_t - \mathcal{F}_{t-1}) + (1-\delta)S_{t-m}$

- $\mathcal{F}_{t+1}$: 1-step ahead forecast.
- $Y_t$: Observation at $t$.
- $T_t$: Trend component at $t$.
- $S_t$: Seasonal component at $t$.

# State Space Models

- Representation of dynamic systems
  - State Variable(s): $X_t$
  - Output Variable(s): $Y_t$
  - State Dynamics: $X_{t+1} \leftarrow \mathrm{X_t}$
  - Observation Process: $Y_t \leftarrow X_t$

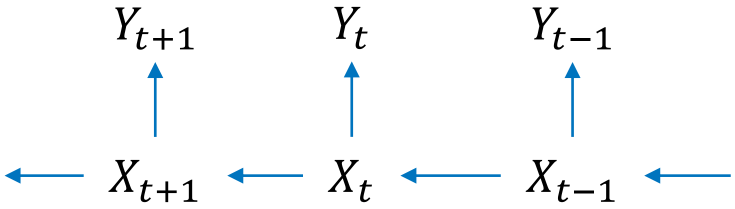



**One Example:**

- $Y_t = AX_t + \epsilon_t$
- $X_t = BX_{t-1} + \eta_t$
- $\epsilon_t$ and $\eta_t$ are uncorrelated

# Smoothing, State Space, Similarities and Differences

- Holt-Winters (additive)
  - $\mathcal{F}_{t+1} = \alpha(Y_t - S_{t-m}) + (1-\alpha)(\mathcal{F}_t + T_t)$
  - $T_{t+1} = \gamma(\mathcal{F}_{t+1} - \mathcal{F}_t) + (1-\gamma)T_t$
  - $S_{t+1} = \delta(Y_t - \mathcal{F}_{t-1}) + (1-\delta)S_{t-m}$
- Representation of dynamic systems




- $T_t$, $S_t$ look like state variables, unobservable. $Y_t$ is observation.
- What is $\mathcal{F}_t$? What are the state dynamics and observation process?

# Exponential Smoothing as State-Space (I)

- Simple Exponential Smoothing (SES)
  - $\mathcal{F}_{t+1} = \alpha Y_t + (1-\alpha)\mathcal{F}_t$ (1)
- State Space Equivalence
  - $Y_t = \ell_{t-1} + \epsilon_t$ (2)
  - $\ell_t = \ell_{t-1} + \alpha\epsilon_t$ (3)
  - Here define $\ell_t = \mathcal{F}_{t+1}$, define $\ell_{t-1} = \mathcal{F}_t$, and obtain $\epsilon_t = Y_t - \ell_{t-1}$ from (2). Substitute symbols in (3) accordingly, we get (1).
  - The smoothing algorithm (1) is to propagate the state dynamics (3).
  - The observation process is (2).
  - $\ell_t$ is the state variable. $Y_t$ is the output variable.

$$Y_{t+1} = \mathcal{F}_{t+1} + \epsilon_{t+1} \qquad Y_t = \mathcal{F}_t + \epsilon_t \qquad Y_{t-1} = \mathcal{F}_{t-1} + \epsilon_{t-1}$$

$$\leftarrow \mathcal{F}_{t+1} = \mathcal{F}_t + \alpha\epsilon_t \leftarrow \mathcal{F}_t = \mathcal{F}_{t-1} + \alpha\epsilon_{t-1} \leftarrow \mathcal{F}_{t-1} = \mathcal{F}_{t-2} + \alpha\epsilon_{t-2} \leftarrow$$

# Exponential Smoothing as State-Space (II)

- Holt's Linear (HL)
  - $\mathcal{F}_{t+1} = \alpha Y_t + (1-\alpha)(\mathcal{F}_t + T_t)$ (1)
  - $T_{t+1} = \gamma(\mathcal{F}_{t+1} - \mathcal{F}_t) + (1-\gamma)T_t$ (2)
- State Space Equivalence
  - $Y_t = \ell_{t-1} + b_{t-1} + \epsilon_t$ (3)
  - $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\epsilon_t$ (4)
  - $b_t = b_{t-1} + \beta\epsilon_t$ (5)
  - Here define $\ell_t = \mathcal{F}_{t+1}$, $\ell_{t-1} = \mathcal{F}_t$, $b_{t-1} = T_t$, and obtain $\epsilon_t = Y_t - \ell_{t-1} - b_{t-1}$ from (3). Substitute symbols in (4) accordingly, we get (1). Similarly, (2) and (5) are equivalent.
  - The smoothing algorithm (1,2) is to propagate the state dynamics (4,5).
  - The observation process is (3).
  - $(\ell_t, b_t)'$ is the state vector. $Y_t$ is the output variable.

# Exponential Smoothing as State-Space (III)

- Holt-Winters (additive) (HW)
  - $\mathcal{F}_{t+1} = \alpha(Y_t - S_{t-m}) + (1-\alpha)(\mathcal{F}_t + T_t)$ (1)
  - $T_{t+1} = \gamma(\mathcal{F}_{t+1} - \mathcal{F}_t) + (1-\gamma)T_t$ (2)
  - $S_{t+1} = \delta(Y_t - \mathcal{F}_{t-1}) + (1-\delta)S_{t-m}$ (3)
- State Space Equivalence
  - $Y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \epsilon_t$ (4)
  - $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\epsilon_t$ (5)
  - $b_t = b_{t-1} + \beta\epsilon_t$ (6)
  - $s_t = s_{t-m} + \theta\epsilon_t$ (7)
  - Here define $\ell_t = \mathcal{F}_{t+1}, \ell_{t-1} = \mathcal{F}_t, b_{t-1} = T_t$, and obtain $\epsilon_t = Y_t - \ell_{t-1} - b_{t-1} - s_{t-m}$ from (3). Substitute symbols in (5) accordingly, we get (1). Similarly, (2) and (6) are equivalent; (3) and (7) are equivalent.
  - The smoothing algorithm (1, 2, 3) is to propagate the state dynamics (5, 6, 7).
  - The observation process is (4).
  - $(\ell_t, b_t, s_t)'$ is the state vector. $Y_t$ is the output variable.

# Exponential Smoothing as State-Space (Biopsy)

- Simple Exponential Smoothing (SES)
  - $Y_t = \ell_{t-1} + \epsilon_t$
  - $\ell_t = \ell_{t-1} + \alpha\epsilon_t$
- Holt's Linear (HL)
  - $Y_t = \ell_{t-1} + b_{t-1} + \epsilon_t$
  - $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\epsilon_t$
  - $b_t = b_{t-1} + \beta\epsilon_t$
- Holt-Winters (additive) (HW)
  - $Y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \epsilon_t$
  - $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\epsilon_t$
  - $b_t = b_{t-1} + \beta\epsilon_t$
  - $s_t = s_{t-m} + \theta\epsilon_t$

State Variables:

- $X_t$: $\ell_t$, $b_t$, and $s_t$

State Dynamics:

- $\ell_t = f_\ell(X_{t-1}, \epsilon_t)$
- $b_t = f_b(X_{t-1}, \epsilon_t)$
- $s_t = f_s(X_{t-1}, \epsilon_t)$

Observation Process:

- $Y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \epsilon_t$

# Exponential Smoothing as State-Space (Expansion)

| | No Seasonal | Additive Seasonal | Multiplicative Seasonal |
|---|---|---|---|
| No Trend | $Y_t = \ell_{t-1} + \epsilon_t$ SES | $Y_t = \ell_{t-1} + s_{t-m} + \epsilon_t$ | $Y_t = \ell_{t-1} s_{t-m} + \epsilon_t$ |
| $+\boldsymbol{b}$ | $Y_t = \ell_{t-1} + b_{t-1} + \epsilon_t$ HL | $Y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \epsilon_t$ HW | $Y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \epsilon_t$ |
| $+\boldsymbol{\phi b}$ | $Y_t = \ell_{t-1} + \phi b_{t-1} + \epsilon_t$ | $Y_t = \ell_{t-1} + \phi b_{t-1} + s_{t-m} + \epsilon_t$ | $Y_t = (\ell_{t-1} + \phi b_{t-1}) s_{t-m} + \epsilon_t$ |
| $\times \boldsymbol{b}$ | $Y_t = \ell_{t-1} b_{t-1} + \epsilon_t$ | $Y_t = \ell_{t-1} b_{t-1} + s_{t-m} + \epsilon_t$ | $Y_t = \ell_{t-1} b_{t-1} s_{t-m} + \epsilon_t$ |
| $\times \boldsymbol{b^{\phi}}$ | $Y_t = \ell_{t-1} b_{t-1}^{\phi} + \epsilon_t$ | $Y_t = \ell_{t-1} b_{t-1}^{\phi} + s_{t-m} + \epsilon_t$ | $Y_t = \ell_{t-1} b_{t-1}^{\phi} s_{t-m} + \epsilon_t$ |
| $\boldsymbol{\varepsilon_t = 1 + \epsilon_t}$ | | Multiplicative Error | |
| No Trend | $Y_t = \ell_{t-1} \varepsilon_t$ | $Y_t = (\ell_{t-1} + s_{t-m}) \varepsilon_t$ | $Y_t = \ell_{t-1} s_{t-m} \varepsilon_t$ |
| $+\boldsymbol{b}$ | $Y_t = (\ell_{t-1} + b_{t-1}) \varepsilon_t$ | $Y_t = (\ell_{t-1} + b_{t-1} + s_{t-m}) \varepsilon_t$ | $Y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} \varepsilon_t$ |
| $+\boldsymbol{\phi b}$ | $Y_t = (\ell_{t-1} + \phi b_{t-1}) \varepsilon_t$ | $Y_t = (\ell_{t-1} + \phi b_{t-1} + s_{t-m}) \varepsilon_t$ | $Y_t = (\ell_{t-1} + \phi b_{t-1}) s_{t-m} \varepsilon_t$ |
| $\times \boldsymbol{b}$ | $Y_t = \ell_{t-1} b_{t-1} \varepsilon_t$ | $Y_t = (\ell_{t-1} b_{t-1} + s_{t-m}) \varepsilon_t$ | $Y_t = \ell_{t-1} b_{t-1} s_{t-m} \varepsilon_t$ |
| $\times \boldsymbol{b^{\phi}}$ | $Y_t = \ell_{t-1} b_{t-1}^{\phi} \varepsilon_t$ | $Y_t = (\ell_{t-1} b_{t-1}^{\phi} + s_{t-m}) \varepsilon_t$ | $Y_t = \ell_{t-1} b_{t-1}^{\phi} s_{t-m} \varepsilon_t$ |

# Exponential Smoothing as State-Space (Recommended)

| | | |
|---|---|---|
| $Y_t = \ell_{t-1} + \epsilon_t$ | $Y_t = \ell_{t-1} + s_{t-m} + \epsilon_t$ | $Y_t = \ell_{t-1} s_{t-m} + \epsilon_t$ |
| $Y_t = \ell_{t-1} + b_{t-1} + \epsilon_t$ | $Y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \epsilon_t$ | $Y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \epsilon_t$ |
| $Y_t = \ell_{t-1} + \phi b_{t-1} + \epsilon_t$ | $Y_t = \ell_{t-1} + \phi b_{t-1} + s_{t-m} + \epsilon_t$ | $Y_t = (\ell_{t-1} + \phi b_{t-1}) s_{t-m} + \epsilon_t$ |
| $Y_t = \ell_{t-1} b_{t-1} + \epsilon_t$ | $Y_t = \ell_{t-1} b_{t-1} + s_{t-m} + \epsilon_t$ | $Y_t = \ell_{t-1} b_{t-1} s_{t-m} + \epsilon_t$ |
| $Y_t = \ell_{t-1} b_{t-1}^{\phi} + \epsilon_t$ | $Y_t = \ell_{t-1} b_{t-1}^{\phi} + s_{t-m} + \epsilon_t$ | $Y_t = \ell_{t-1} b_{t-1}^{\phi} s_{t-m} + \epsilon_t$ |

| | | |
|---|---|---|
| $Y_t = \ell_{t-1} \varepsilon_t$ | $Y_t = (\ell_{t-1} + s_{t-m}) \varepsilon_t$ | $Y_t = \ell_{t-1} s_{t-m} \varepsilon_t$ |
| $Y_t = (\ell_{t-1} + b_{t-1}) \varepsilon_t$ | $Y_t = (\ell_{t-1} + b_{t-1} + s_{t-m}) \varepsilon_t$ | $Y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} \varepsilon_t$ |
| $Y_t = (\ell_{t-1} + \phi b_{t-1}) \varepsilon_t$ | $Y_t = (\ell_{t-1} + \phi b_{t-1} + s_{t-m}) \varepsilon_t$ | $Y_t = (\ell_{t-1} + \phi b_{t-1}) s_{t-m} \varepsilon_t$ |
| $Y_t = \ell_{t-1} b_{t-1} \varepsilon_t$ | $Y_t = (\ell_{t-1} b_{t-1} + s_{t-m}) \varepsilon_t$ | $Y_t = \ell_{t-1} b_{t-1} s_{t-m} \varepsilon_t$ |
| $Y_t = \ell_{t-1} b_{t-1}^{\phi} \varepsilon_t$ | $Y_t = (\ell_{t-1} b_{t-1}^{\phi} + s_{t-m}) \varepsilon_t$ | $Y_t = \ell_{t-1} b_{t-1}^{\phi} s_{t-m} \varepsilon_t$ |

# Exponential Smoothing as State-Space ($h$-ahead Forecast)

| | | |
|---|---|---|
| $\hat{y}_{t+h} = \ell_t$ | $\hat{y}_{t+h} = \ell_t + s_{t-m+h_m}$ | |
| $\hat{y}_{t+h} = \ell_t + hb_t$ | $\hat{y}_{t+h} = \ell_t + hb_t + s_{t-m+h_m}$ | |
| $\hat{y}_{t+h} = \ell_t + \phi_h b_t$ | $\hat{y}_{t+h} = \ell_t + \phi_h b_t + s_{t-m+h_m}$ | |
| | | |
| | | |

| | | |
|---|---|---|
| $\hat{y}_{t+h} = \ell_t$ | $\hat{y}_{t+h} = \ell_t + s_{t-m+h_m}$ | $\hat{y}_{t+h} = \ell_t s_{t-m+h_m}$ |
| $\hat{y}_{t+h} = \ell_t + hb_t$ | $\hat{y}_{t+h} = \ell_t + hb_t + s_{t-m+h_m}$ | $\hat{y}_{t+h} = (\ell_t + hb_t)s_{t-m+h_m}$ |
| $\hat{y}_{t+h} = \ell_t + \phi_h b_t$ | $\hat{y}_{t+h} = \ell_t + \phi_h b_t + s_{t-m+h_m}$ | $\hat{y}_{t+h} = (\ell_t + \phi_h b_t)s_{t-m+h_m}$ |
| $\hat{y}_{t+h} = \ell_t b_t^h$ | | $\hat{y}_{t+h} = \ell_t b_t^h s_{t-m+h_m}$ |
| $\hat{y}_{t+h} = \ell_t b_t^{\phi_h}$ | | $\hat{y}_{t+h} = \ell_t b_t^{\phi_h} s_{t-m+h_m}$ |

$\phi_h$ is a function of $\phi$ and $h$. $h_m$ is a function of $h$ and $m$.

# Exponential Smoothing as State-Space ($h$-ahead Forecast)

| | | |
|---|---|---|
| $\ell_t$: Constant | $\ell_t + s_{t-m+h_m}$: Seasonal Pattern | |
| $\ell_t + hb_t$: Linear Trend | Linear Trend + Seasonal Pattern | |
| $\ell_t + \phi_h b_t$: Damped Trend | Damped Trend + Seasonal Pattern | |
| | | |
| | | |

| | | |
|---|---|---|
| $\ell_t$: Constant | $\ell_t + s_{t-m+h_m}$: Seasonal Pattern | $\ell_t s_{t-m+h_m}$: Seasonal Pattern |
| $\ell_t + hb_t$: Linear Trend | Linear Trend + Seasonal Pattern | Linear Trend × Seasonal Pattern |
| $\ell_t + \phi_h b_t$: Damped Trend | Damped Trend + Seasonal Pattern | Damped Trend × Seasonal Pattern |
| $\ell_t b_t^h$: Non-linear Trend | | Non-linear Trend × Seasonal Pattern |
| $\ell_t b_t^{\phi_h}$: Non-linear Trend | | Non-linear Trend × Seasonal Pattern |