# Variances of Classicial Variance Component Estimators

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For these calculations, we will assume there are three factors: Factor $A$, Factor $B$, and Factor $C$, each with $a, b$, and $c$ levels, respectively. There are also $r$ replicates of the entire design. The variance estimates are computed using the classical expected mean squares approach. The variance of the estimators are considered under three different models: fully crossed design (includes interactions), main effects only, and the fully nested design, where Factor $C$ is nested within Factors $A$ and $B$ and Factor $B$ is nested within Factor $A$.

## 1 Fully Crossed Design

Under a fully crossed design, the model is:

$$Y_{ijkl} = \mu + A_i + B_j + C_k + (AB)_{ij} + (AC)_{ik} + (BC)_{jk} + (ABC)_{ijk} + \varepsilon_{ijkl} \tag{1}$$

where

$$A_i \sim \text{Normal}(0, \sigma_A^2) \tag{2}$$

$$B_i \sim \text{Normal}(0, \sigma_B^2) \tag{3}$$

$$C_i \sim \text{Normal}(0, \sigma_C^2) \tag{4}$$

$$(AB)_{ij} \sim \text{Normal}(0, \sigma_{AB}^2) \tag{5}$$

$$(AC)_{ik} \sim \text{Normal}(0, \sigma_{AC}^2) \tag{6}$$

$$(BC)_{jk} \sim \text{Normal}(0, \sigma_{BC}^2) \tag{7}$$

$$(ABC)_{ijk} \sim \text{Normal}(0, \sigma_{ABC}^2) \tag{8}$$

$$\varepsilon_{ijkl} \sim \text{Normal}(0, \sigma_E^2) \tag{9}$$

$$\tag{10}$$

The full suite of variance estimates is given as:

$$\widehat{\sigma}_A^2 = \frac{MSA - MSAB - MSAC + MSABC}{bcr} \tag{11}$$

$$\widehat{\sigma}_B^2 = \frac{MSB - MSAB - MSBC + MSABC}{acr} \tag{12}$$

$$\widehat{\sigma}_C^2 = \frac{MSC - MSAC - MSBC + MSABC}{abr} \tag{13}$$

$$\widehat{\sigma}_{AB}^2 = \frac{MSAB - MSABC}{cr} \tag{14}$$

$$\widehat{\sigma}_{AC}^2 = \frac{MSAC - MSABC}{br} \tag{15}$$

$$\widehat{\sigma}_{BC}^2 = \frac{MSBC - MSABC}{ar} \tag{16}$$

$$\widehat{\sigma}_{ABC}^2 = \frac{MSABC - MSE}{r} \tag{17}$$

$$\widehat{\sigma}_E^2 = MSE \tag{18}$$

The MS terms are the mean squares, which are distributed as follows

$$MSA \sim (bcr\sigma_A^2 + cr\sigma_{AB}^2 + br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_E^2)\frac{\chi_{a-1}^2}{a-1} \tag{19}$$

$$MSB \sim (acr\sigma_B^2 + cr\sigma_{AB}^2 + ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_E^2)\frac{\chi_{b-1}^2}{b-1} \tag{20}$$

$$MSC \sim (abr\sigma_C^2 + br\sigma_{AC}^2 + ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_E^2)\frac{\chi_{c-1}^2}{c-1} \tag{21}$$

$$MSAB \sim (cr\sigma_{AB}^2 + r\sigma_{ABC}^2 + \sigma_E^2)\frac{\chi_{(a-1)(b-1)}^2}{(a-1)(b-1)} \tag{22}$$

$$MSAC \sim (br\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma_E^2)\frac{\chi_{(a-1)(c-1)}^2}{(a-1)(c-1)} \tag{23}$$

$$MSBC \sim (ar\sigma_{BC}^2 + r\sigma_{ABC}^2 + \sigma_E^2)\frac{\chi_{(b-1)(c-1)}^2}{(b-1)(c-1)} \tag{24}$$

$$MSABC \sim (r\sigma_{ABC}^2 + \sigma_E^2)\frac{\chi_{(a-1)(b-1)(c-1)}^2}{(a-1)(b-1)(c-1)} \tag{25}$$

$$MSE \sim \sigma_E^2\frac{\chi_{abc(r-1)}^2}{abc(r-1)} \tag{26}$$

Using the above expressions and the independence of the MS terms, we can compute the variances of the variance component estimators. For the purposes of easy illustration, we'll only present the variances for selected variance components. It should be straightforward to extrapolate to other similar variance components.

To start, the variance of $\widehat{\sigma}_A^2$ is computed, after some algebra and simplification, as

$$\mathrm{Var}(\widehat{\sigma}_A^2) = \frac{2}{a-1}\Bigg[\sigma_A^4 + \frac{\sigma_{AB}^4}{b(b-1)} + \frac{\sigma_{AC}^4}{c(c-1)} + \frac{\sigma_{ABC}^4}{b(b-1)c(c-1)} + \frac{\sigma_E^4}{r^2b(b-1)c(c-1)} \tag{27}$$

$$+\frac{2\sigma_A^2\sigma_{AB}^2}{b} + \frac{2\sigma_A^2\sigma_{AC}^2}{c} + \frac{2\sigma_A^2\sigma_{ABC}^2}{bc} + \frac{2\sigma_A^2\sigma_E^2}{bcr} + \frac{2\sigma_{AB}^2\sigma_{AC}^2}{bc} \tag{28}$$

$$+\frac{2\sigma_{AB}^2\sigma_{ABC}^2}{b(b-1)c} + \frac{2\sigma_{AB}^2\sigma_E^2}{b(b-1)cr} + \frac{2\sigma_{AC}^2\sigma_{ABC}^2}{bc(c-1)} + \frac{2\sigma_{AC}^2\sigma_E^2}{bc(c-1)r} + \frac{2\sigma_{ABC}^2\sigma_E^2}{b(b-1)c(c-1)r}\Bigg] \tag{29}$$

The variance of the other main variance component estimators are similar.