Box 3.4 Calculating partial covariances and correlations

Given a sample covariance matrix **S**, the inverse of this matrix is called the *concentration* matrix, **C**. The negative of the off-diagonal elements c_{ij} give the partial covariance between variables i and j, conditional on (holding constant) all the other variables included in the matrix. This gives an easy way of estimating partial covariances and partial correlations of any order. To get the partial covariance between variables X and Y conditional on a set of other variables **Q**, simply create a covariance matrix in which the only variables are X, Y and the remaining variables in **Q**. Invert the matrix, and then this partial covariance is the negative of the element in the row pertaining to X and the column pertaining to Y – namely $-c_{XY}$. The partial correlation between X and Y is given by

$$r_{X,Y|Q} = \frac{-c_{XY}}{\sqrt{c_{XX} \cdot c_{YY}}}$$

The partial correlation between two variables conditioned on n other variables is said to be a *partial correlation of order n*. The unconditional correlation coefficient is simply a partial correlation of order 0. Some texts give recursion formulae for partial correlations of various orders, although partials of higher orders are very tedious to calculate by such means. For instance, the formula for a partial correlation of order 1 between X and Y, conditional on Z, is

$$\rho_{X,Y|Z} = \frac{\rho_{XY} - \rho_{XZ}\rho_{YZ}}{\sqrt{\left(1-\rho_{XZ}^2\right)\left(1-\rho_{YZ}^2\right)}}$$

As an example, consider the following causal graph: W→X→Z→Y. 100 independent (W,X,Y,Z) observations were generated according to structural equations with all path coefficients equal to 0.5 and the variances of all four variables equal to 1.0. Here is the sample covariance matrix:

	W	X	Y	Z
W	1.43347870	−0.75265627	−0.06269845	0.10179918
X	−0.75265627	1.52762094	−0.53911722	−0.03777874
Y	−0.06269845	−0.53911722	1.71116716	−0.90033856
Z	0.10179918	−0.03777874	−0.90033856	1.73196991

The inverse of the matrix (rounded to the nearest 100th) obtained by extracting only the elements of the covariance matrix pertaining to W, X and Y is

	W	X	Y
W	1.43	−0.75	−0.01
X	−0.75	1.53	−0.56
Y	−0.01	−0.56	1.24

The partial correlation between W and Y, conditional on X, is

$$r_{WY|X} = \frac{(-1) - 0.01}{\sqrt{1.43 \cdot 1.24}} = 0.0075$$

The same method can be used to obtain partial Spearman partial correlations, by simply ranking the variables as described in Box 3.2 and then proceeding in the same way as for Pearson partial correlations.