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Report No. 8

Studies in Quality Improvement: Minimizing Transmitted Variation by Parameter Design

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February 1986

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ABSTRACT

The reproducibility of a quality characteristic of a manufactured product inevitably depends on the amount of variation in the components that go to make the product. This is because variations in the components are "transmitted" to the quality characteristic. Careful choice of the nominal values of the components' characteristics can minimize this transmission. This process of choice is an example of what professor Genichi Taguchi calls *parameter design*.

In this paper the design of a Wheatstone Bridge circuit used by Taguchi to illustrate the orthogonal array method of parameter design is discussed and reconsidered. It is pointed out that for this particular example the orthogonal array method does not yield the overall optimal solution. Simpler methods are discussed, in particular the use of nonlinear programming. A list of related problems is presented.

KEYWORDS: *Parameter design; Transmission of error; Signal-to-noise ratio; Orthogonal arrays; Nonlinear programming; Wheatstone Bridge.*

This work was sponsored by the National Science Foundation under grant DMS-8420968, by the United States Army under contract DAAG29-80-C-0041, and by the Vilas Trust of the University of Wisconsin-Madison.

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In their book *Introduction to Off-Line Quality Control* (1979), Taguchi and Wu discuss a number of important and interesting problems concerned with the improvement of quality and efficiency of products and processes. Many of these ideas are novel and valuable and have only recently begun to receive attention from quality technologists and statisticians (see for example Kackar, 1985). In this paper we discuss one such problem they call *parameter design.*

THE PARAMETER DESIGN PROBLEM

The authors illustrate the problem using a well known electrical circuit called the Wheatstone Bridge, used for the determination of an unknown resistance p_0. It is supposed that this circuit is to be used in some manufactured product and that design specifications for its components are needed such that optimal precision is achieved in the determination of p_0. The circuit is shown diagramatically in Figure 1. The components A, C, D, and F are fixed resistances; E is a fixed battery voltage; and X is an ammeter reading. If the variable resistance B is adjusted so that there is no flow of current through the ammeter, an estimate y of the unknown resistance p_0 can be calculated from the formula

$$y = \frac{BD}{C}. \qquad (1)$$

More generally if a current X is flowing through the ammeter, the relation becomes

$$y = \frac{BD}{C} - \frac{X}{C^2 E}\left[A(C+D) + D(B+C)\right] \cdot \left[B(C+D) + F(B+C)\right] \qquad (2)$$

by which y is related to all the factors A, B, C, D, E, F, and X.

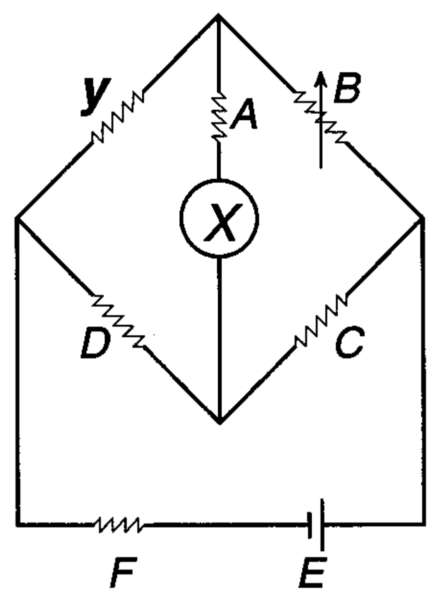

Figure 1. *Diagram of Wheatstone Bridge.*

Now consider a particular sample of components *A, B, C, D, E, F, X*, brought together to manufacture a single complete circuit. Each component characteristic will deviate slightly from its nominal value and these deviations will *transmit* errors into *y*. However, when the product is being designed, the nominal values for the fixed resistances – *A, C, D*, and *F*, and the fixed battery voltage *E* – are *at choice*, and the error transmission will be different depending on which choice is made. The parameter design question is: how should the nominal values of these "design" variables be chosen? How should the product be *designed* so that the error transmitted to *y* is small?

The authors point out that from such a study it is often possible to achieve reduced error transmission using cheaper components even when these components are individually more variable than more expensive ones.

Although error transmission has frequently been discussed in other contexts (see for example Deming, 1964; Ku, 1969; Box, Hunter, and Hunter, 1978), this very important application seems previously to have received very little attention outside Japan. Indeed, in the United States and Europe the techniques of quality control have been largely concerned with process surveillance and sampling inspection. Except in the chemical industry, the important role that statistical methods and particularly experimental design should play in *product design* and *process design*, seems to have been largely overlooked.

In what follows, we employ the terminology *design factors** for those components whose nominal levels are at our choice in designing the product (*A, C, D, E, F* in this example). Also, following Taguchi the term *error factors* will refer to all the components that transmit error to *y* (*A, B, C, D, E, F, X* in this example). Thus, for instance, *A* is both a design factor and an error factor, but *B* and *X* are error factors only.

It is supposed that the range of possible values allowable for the nominal value of each design factor is known. Typically this range is very wide. For instance, the values used in the present example, which are shown in Table 1(a), employ a ratio of (highest value)/(lowest value) of 25. It is convenient to refer to the allowable region so defined as the *design region R*. The coefficient of variation is also supposed known for each design factor. This measures the variation to be expected about the nominal value from one item to the next. These coefficients of variation are typically small compared with the allowable ranges. Thus in this example (see Table 1(b)), except for factor *E*, the coefficients of variation* for the design factors are all less than 0.25%.

Table 1(a)

Levels of Design Factors in the Inner Array.

FACTORS	*1ST LEVEL* (–)	*2ND LEVEL* (+)	*3RD LEVEL* (+)
A (ohms)	20	100	500
C (ohms)	2	10	50
D (ohms)	2	10	50
E (volts)	1.2	6	30
F (ohms)	2	10	50

Table 1(b)

Levels of Error Factors in the Outer Arrays.

FACTORS	*1ST LEVEL* (–)	*2ND LEVEL* (+)	*3RD LEVEL* (+)	*IMPLIED CV*
A (%)	–0.3	0	0.3	0.24%
B (%)	–0.3	0	0.3	0.24%
C(%)	–0.3	0	0.3	0.24%
D (%)	–0.3	0	0.3	0.24%
E(%)	–5.0	0	5.0	4.08%
F (%)	–0.3	0	0.3	0.24%
X (mA)	–0.2	0	0.2	[s.d. = 0.16 mA)

As the objective function to be maximized, Taguchi and Wu chose the measure $10\log_{10}(SN_T)$, where SN_T is a "signal-to-noise ratio" they define as

$$SN_T = \left\{ \frac{\bar{y}^2}{s_y^2} - \frac{1}{36} \right\}. \qquad (3)$$

Maximizing SN_T is equivalent to maximizing $SN = \bar{y}^2/s_y^2$; henceforward the notation "*SN*" will denote this latter quantity.

* Taguchi uses the terminology *control factors* and *parameters* to describe these variables, but we find the phrase *design factors* to be more descriptive.

* The ranges of variation for each error factor (Table 1(b)) were calculated so that the standard deviations of the implied 3-point discrete distributions equaled the standard deviations of the assumed continuous distributions; thus, for example, for factor *A* the error factor levels are 0 and $\pm\sigma_A\sqrt{3/2}$.

A SOLUTION EMPLOYING ORTHOGONAL ARRAYS

Taguchi and Wu's solution was based on a double application of orthogonal array designs as follows. A layout that they called the *inner array* was chosen for the design factors – for this example combinations of the three levels of Table 1(a) for each of the five design factors (A, C, D, E, F) were employed using the orthogonal array L_{36} shown in Table 2. A, C, D, E, and F were assigned to columns 1, 3, 4, 5, and 6 respectively. The design was thus made to span the space R of the whole design region of interest. At each of these inner array points an arrangement called

Table 2.
Orthogonal Array L_{36}. The appended column shows the values of $10 log_{10}(SN_T)$ obtained by Taguchi at the 36 inner array points.

	COLUMN NUMBER													
Run No.	1	2	3	4	5	6	7	8	9	10	11	12	13	*Taguchi's Criterion*
1	–	–	–	–	–	–	–	–	–	–	–	–	–	32.2
2	0	0	0	0	0	0	0	0	0	0	0	0	–	26.7
3	+	+	+	+	+	+	+	+	+	+	+	+	–	15.9
4	–	–	–	–	0	0	0	0	+	+	+	+	–	36.4
5	0	0	0	0	+	+	+	+	–	–	–	–	–	28.6
6	+	+	+	+	–	–	–	–	0	0	0	0	–	7.2
7	–	–	0	+	–	0	+	+	–	0	0	+	–	16.5
8	0	0	+	–	0	+	–	–	0	+	+	–	–	13.0
9	+	+	–	0	+	–	0	0	+	–	–	0	–	28.0
10	–	–	+	0	–	+	0	+	0	–	+	0	–	15.0
11	0	0	–	+	0	–	+	–	+	0	–	+	–	16.4
12	+	+	0	–	+	0	–	0	–	+	0	–	–	25.5
13	–	0	+	–	+	0	–	+	+	0	–	0	0	43.8
14	0	+	–	0	–	+	0	–	–	+	0	+	0	–8.3
15	+	–	0	+	0	–	+	0	0	–	+	–	0	14.6
16	–	0	+	0	–	–	+	0	+	+	0	–	0	29.0
17	0	+	–	+	0	0	–	+	–	–	+	0	0	6.9
18	+	–	0	–	+	+	0	–	0	0	–	+	0	14.7
19	–	0	–	+	+	+	–	0	0	–	0	+	0	21.5
20	0	+	0	–	–	–	0	+	+	0	+	–	0	17.4
21	+	–	+	0	0	0	+	–	–	+	–	0	0	14.0
22	–	0	0	+	+	–	0	–	–	+	+	0	0	46.5
23	0	+	+	–	–	0	+	0	0	–	–	+	0	5.5
24	+	–	–	0	0	+	–	+	+	0	0	–	0	–8.2
25	–	+	0	–	0	+	+	–	+	–	0	0	+	27.3
26	0	–	+	0	+	–	–	0	–	0	+	+	+	43.4
27	+	0	–	+	–	0	0	+	0	+	–	–	+	–20.9
28	–	+	0	0	0	–	–	+	0	+	–	+	+	44.1
29	0	–	+	+	+	0	0	–	+	–	0	–	+	39.3
30	+	0	–	–	–	+	+	0	–	0	+	0	+	–17.0
31	–	+	+	+	0	+	0	0	–	0	–	–	+	23.0
32	0	–	–	–	+	–	+	+	0	+	0	0	+	44.2
33	+	0	0	0	–	0	–	–	+	–	+	+	+	–0.9
34	–	+	–	0	+	0	+	–	0	0	+	–	+	43.4
35	0	–	0	+	–	+	–	0	+	+	–	0	+	–7.7
36	+	0	+	–	0	–	0	+	–	–	0	+	+	8.0

an *outer array* was now employed in which the error factors were allowed to vary over the small ranges listed in Table 1(b). In this example the outer arrays were again L_{36} designs run in the seven error factors *A, B, C, D, E, F, X*, which were assigned respectively to columns 1 through 7. The full layout, illustrated conceptually in Figure 2, thus consisted of 36 × 36 = 1296 points*.

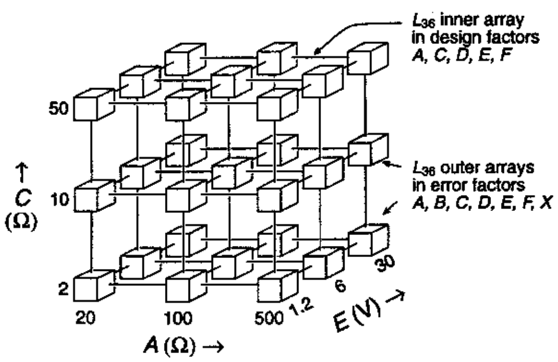

Figure 2. *Conceptual illustration of how outer arrays are arranged at each inner array point.*

Using Equation (2) the *y*'s could now be calculated at each of the 1296 points, and consequently an SN_T ratio was computed for each of the 36 inner array points. The 36 resulting values of $10\log_{10}(SN_T)$, shown in the last column of Table 2, were then subjected to an analysis of variance procedure and marginal averages were calculated for the (–), (0), and (+) levels of each design factor. The marginal averages are shown in Table 3 and are plotted in Figure 3. By inspection of these marginal averages the authors reached the conclusion that the objective function was maximized for the levels $A_{(-)}$, $C_{(+)}$, $D_{(0)}$, $E_{(+)}$, and $F_{(-)}$.

Table 3.
Marginal averages of $10\log_{10}(SN_T)$ *at the inner array levels of each design factor.*

FACTOR	1ST LEVEL (–)	2ND LEVEL (0)	3RD LEVEL (+)
A	31.56	18.78	6.73
C	14.56	21.10	21.42
D	20.91	21.24	14.93
E	5.66	18.52	32.89
F	27.58	19.68	9.81

Direct calculation, however, shows that this is not the value that maximizes SN_T and that a value some 6% larger** is obtained with levels $A_{(-)}$, $C_{(0)}$, $D_{(0)}$, $E_{(+)}$, and $F_{(-)}$, while if non-integer values are considered an even higher value is possible.

The situation can be readily appreciated by graphical examination of the objective function $10\log_{10}(SN_T)$ in the space of the design factors. This turns out to be roughly linear in *A*, *E*, and *F*, resulting in optimal values for these factors being at the extremes $A_{(-)}$, $E_{(+)}$, and $F_{(-)}$, for fixed values of *C* and *D*. It remains therefore to find the optimal values of *C* and *D* on the *CD* plane obtained when *A*, *E*, and *F* are set at these optimal levels. Figure 4 shows contours of $10\log_{10}(SN_T)$ on this plane with the orthogonal array solution marked *OA*. Also marked is the point at which the maximum is actually obtained. It will be seen that the contours about the maximum are obliquely oriented in a manner associated with a large interaction between these variables. Such an interaction is to be expected because of the dominant role played by the ratio *D/C* in Equation (2). In the presence of such an interaction the marginal graphs of Figure 3 will of course be misleading and indeed the maximum is seen to be located a considerable distance from that found by the orthogonal array method.

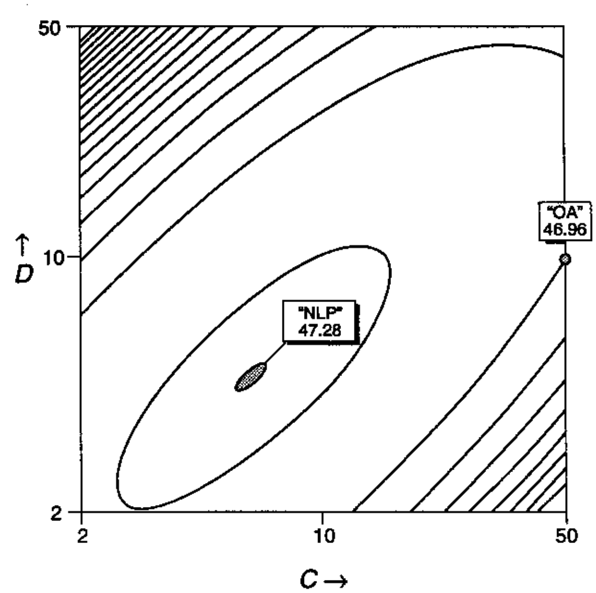

Figure 4. *Contours of* $10\log_{10}(SN_T)$ *in the CD plane. A, E, and F are set at their optimal levels.*

* In the outer arrays, the central (0) level of *B* is set at $B = p_0C/D = 2C/D$ where, following Taguchi, $p_0 = 2$ is the target resistance for which maximum precision is to be achieved. The central level of *X* is 0 by definition.

** This comparison is in the SN_T metric rather than $10\log_{10}(SN_T)$ to show directly the relative signal-to-noise levels.

Although in this particular example the extent by which the orthogonal array solution falls short of the maximum is not particularly large, the fact that it can readily do so calls for investigation.

A MORE GENERAL FORMULATION OF THE PROBLEM

For convenience in what follows we will write $H = \log_{10}(SN)$ and consider the equivalent problem of minimizing H. The quantities that appear in the numerator and denominator of the signal-to-noise ratio SN are calculable from (2). The numerator y is obtained directly; the denominator may be obtained from the usual error transmission formula employing the first partial derivatives of y with respect to the error factors. Thus H is, in principle, a known function of the levels of the design factors, and the problems is one of minimizing a known function within a given known region.

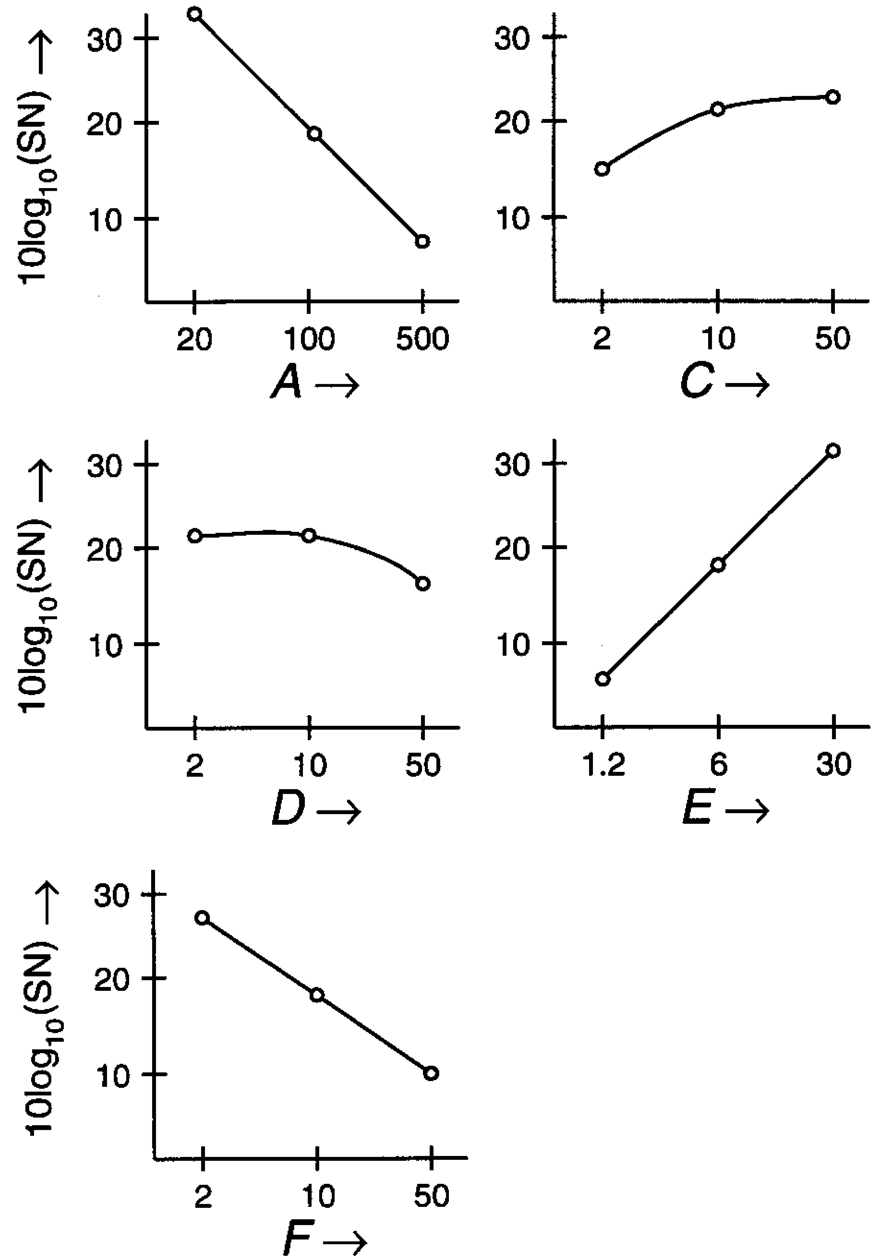

Figure 3. *Plots of the marginal averages from Table 3.*

Specifically, if we use x_1, x_2, etc. to denote the factors, the problem consists of defining:

(a) a d-dimensional design vector $\underset{\sim}{x}_d$ whose elements can be chosen at will within some known design region R;

(b) a k-dimensional error vector $\underset{\sim}{x}_e$ some or all of whose elements may be identical with those of $\underset{\sim}{x}_d$ and whose *known* covariance matrix is Σ_e. Typically it will be assumed that the elements of $\underset{\sim}{x}_e$ vary independently so that Σ_e is diagonal with i^{th} diagonal element σ_i^2; and

(c) an objective function H that, given Σ_e, can be computed and that in some acceptable sense measures the relative variation in a quantity of interest y whose functional relationship $y = f(\underset{\sim}{x}_e)$ to $\underset{\sim}{x}_e$ is *known.*

The object is then to minimize the known objective function H within the known region R. The only question is: what are the relative merits of various methods of achieving this?

THE WHEATSTONE BRIDGE EXAMPLE IN THE MORE GENERAL CONTEXT

Before proceeding further we illustrate this general formulation with the Wheatstone Bridge example previously considered. Taguchi and Wu imply by their choice of levels in Tables 1(a) and 1(b) that the appropriate metric for measuring *A, B, C, D, E,* and *F* is the log. Thus logged values for these inputs will be used, and we will write $x_1, x_2, \ldots, x_6$ to denote the factors *A, B, C, D, E,* and *F* after being logged, scaled, and centered to cover the interval -1 to $+1$. The error standard deviations $\sigma_1, \sigma_2, \ldots, \sigma_7$ of the x variables will be proportional to the coefficients of variation of the original error factors and will not depend on $\underset{\sim}{x}$. In this transformed metric, Equation (2) can be rewritten

$$y_{\underset{\sim}{x}} = f\left(\underset{\sim}{x}\right) = f(x_1, x_2, \ldots, x_7) \qquad (4)$$

and as indicated above it is possible to compute Taguchi's SN_T ratio using outer arrays at any point $\underset{\sim}{x}$ and to seek its maximum. However, it is worthwhile to simplify further as follows.

The reciprocal of the signal-to-noise ratio SN is the square of the coefficient of variation of y and for small relative errors such as are encountered in this and similar examples, the coefficient of variation is to a very good approximation equal to the standard deviation of $\ln y$. Thus the problem is essentially that of minimizing $V_{\underset{\sim}{x}}$, the variance of $Y_{\underset{\sim}{x}} = \ln y_{\underset{\sim}{x}}$.

Taking logs on both sides of (4)

$$Y_{\underset{\sim}{x}} = \ln f\left(\underset{\sim}{x}\right) = F\left(\underset{\sim}{x}\right), \qquad (5)$$

and given the small error variations in the x_i's contemplated in this example, the variance of $Y_{\underset{\sim}{x}}$ is closely approximated by the first-order error transmission formula*

$$V_{\underset{\sim}{x}} = d_1^2\left(\underset{\sim}{x}\right)\sigma_1^2 + d_2^2 + \ldots + d_7^2\left(\underset{\sim}{x}\right)\sigma_7^2 \qquad (6)$$

where

$$d_i\left(\underset{\sim}{x}\right) = \left.\frac{\partial F}{\partial x_i}\right|_{\underset{\sim}{x}} \quad i = 1.2, \ldots, 7. \qquad (7)$$

The problem formally stated is thus to minimize the known function $V_{\underset{\sim}{x}}$ or equivalently $H'_{\underset{\sim}{x}} = 10\log_{10} V_{\underset{\sim}{x}}$ within the known design region R' defined by

$$-1 \le x_i \le +1 \quad i = 1, 3, 4, 5, 6.$$

The d's could be determined by direct differentiation, but in practice they are much more conveniently found numerically as is normally done for example in nonlinear least squares calculations. In what follows, the i^{th} partial derivative $d_i\left(\underset{\sim}{x}\right)$ at some point $\underset{\sim}{x} = (x_1, x_2, \ldots, x_k)$ is computed as the divided difference of $Y = F(\underset{\sim}{x})$ when the level of the i^{th} factor is raised from x_i to $x_i + \delta_i$ and all the other factors are kept fixed**. Thus to determine the derivatives for k

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* Any transmitted variation function $V_{\underset{\sim}{x}}$ can be written in the form (6) whether the original response function $f(\underset{\sim}{x})$ is explicit or implicit (as is the case in the Wheatstone Bridge problem); see Appendix 1 for further discussion of this special case.

** In some cases it might be worthwhile to put in a nonlinearity check using an additional k points, the i^{th} of which is at $(x_1, x_2, \ldots, x_i - \delta_i, \ldots, x_k)$ as is suggested for example in Box, Hunter and Hunter (1978). However, if approximate linearity breaks down over the small increments δ_i normally contemplated, the global situation over the *whole* design region might be very complicated indeed and serious difficulties might be expected whatever the method of approach.

error factors, only $k + 1$ runs (8 in this example) are needed. Thus:

$$d_i(\underset{\sim}{x}) \approx \frac{F(x_1, x_2, \ldots, x_i + \delta_i, \ldots, x_7) - F(\underset{\sim}{x})}{\delta_i}. \quad (8)$$

The arrangement for the case $k = 3$ is illustrated in Figure 5 with the δ_i made proportional to the σ_i by for example using the increments implied by Table 1(b). Then $\delta_i = \alpha\delta_i$, where α is some constant, and if $Y_0, Y_1, Y_2, \ldots, Y_k$ are the values $Y = F(\underset{\sim}{x})$ at the various design points, equation (6) becomes simply

$$V_{\underset{\sim}{x}} \approx \frac{1}{\alpha^2}\left\{(Y_1 - Y_0)^2 + (Y_2 - Y_0)^2 + \ldots + (Y_k - Y_0)^2\right\}. \quad (9)$$

Thus up to an additive constant the objective function $H'_{\underset{\sim}{x}}$ for this example may be computed from

$$H'_{\underset{\sim}{x}} = 10\log_{10} V_{\underset{\sim}{x}}$$
$$= 10\log_{10}\left\{(Y_1 - Y_0)^2 + (Y_2 - Y_0)^2 + \ldots + (Y_7 - Y_0)^2\right\} \quad (10)$$
$$+ \text{const.}$$

Notice that this mode of calculation is not peculiar to this example. Any other criterion that involved the level and the variance of y could be computed similarly.

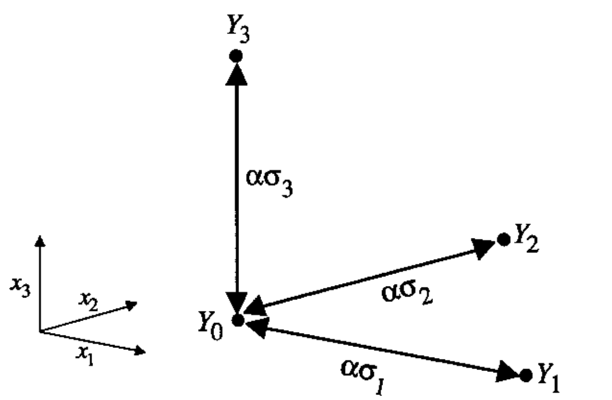

Figure 5. *An arrangement for determining* $V_{\underset{\sim}{x}}$ *when* $k = 3$.

For comparison we have used equation (10)* with $\alpha = 10^{-3}$ to compute $H'_{\underset{\sim}{x}}$ for all 36 points in the L_{36} inner array (see Appendix 2). We find that our results, based on eight-point layouts, closely approximate the values obtained by Taguchi and Wu using 36-point outer arrays. The correlation between the two sets of values is greater than .9999.

In its general form the problem is seen to be an example of the *constrained optimization* or *nonlinear programming* problem of optimizing a known function within a given space, which has received considerable attention over many years (see for example Kuhn and Tucker, 1951; Whittle, 1971; and McCormick, 1983). Many efficient computer subroutines to make such calculations are available. In particular we have used both the IMSL program ZXMWD that employs Fletcher's quasi-Newton algorithm VA10A (Fletcher, 1972), and the NAGLIB program E04CCF that employs the simplex method of Nelder and Mead (1964). All that is required in addition is a subroutine to compute $V_{\underset{\sim}{x}}$ using (9), or equivalently $H'_{\underset{\sim}{x}}$ from (10), for any $\underset{\sim}{x}$. A standard "packaged" program for the parameter design problem may readily be written and might, for example, include the features shown in Figure 6.

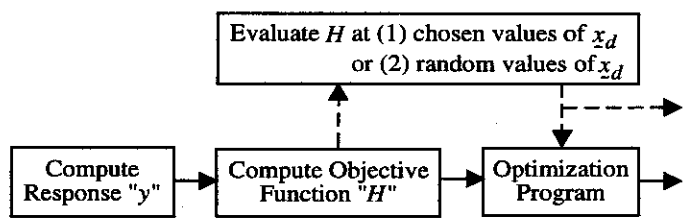

Figure 6. *Possible features of a standard computer program for parameter design.*

For the Wheatstone Bridge example our computations led to the correct optimal value indicated in Figure 4 as the nonlinear programming solution (*NLP*).

CONSIDERATION OF SOME RELEVANT ISSUES

We here considered some issues in the constrained optimization problem that arise from parameter design applications. In general the degree of difficulty associated with minimizing a function $H(\underset{\sim}{x})$ within a region R must depend on (a) the complexity and dimension of R and (b) the complexity of $H(\underset{\sim}{x})$.

* Equation (2) has the peculiarity that at $X = 0$ it degenerates to Equation (1) that does not contain A, E, F, and X. However, all the partial derivatives still exist and are continuous in X.

THE DESIGN REGION R

So far as the design region R is concerned, it appears that in most applications of parameter design R would be defined by a series of simple ranges as in Table 1(a), and this would not present special difficulties. Also the region R is seldom unchangeably fixed. Experience with response surface studies shows that when a direction of advance impinges on a constraint defining R, the possibility is considered of changing the system so that R may be extended in the indicated directions (see Appendix 3).

NATURE OF $H(\underset{\sim}{x})$

If the objective function $H(\underset{\sim}{x})$ might be of any degree of complexity whatever, then any method of optimization could be defeated. Much depends on the extent to which smoothness properties of the function can be relied upon. For example, in a finite time no method could find the minimum of a function which had everywhere a constant positive value except at one point where it was zero. So we need to consider in the present context what special information, if any, is available about this matter.

For the parameter design problem the objective function $V_{\underset{\sim}{x}}$ has some especially helpful properties. In particular from the Equation (6), the function $V_{\underset{\sim}{x}}$ is positive except at a point $\underset{\sim}{x}_0$ where $d_1(\underset{\sim}{x}_0)= d_2(\underset{\sim}{x}_0)= \ldots = d_k(\underset{\sim}{x}_0)=0$. Thus if there exists such a point x_0 internal to the region, it must be a minimum for $V_{\underset{\sim}{x}}$ (although not necessarily a unique minimum) and it must be a stationary point of the response function $F(\underset{\sim}{x})$. It is also easy to show that if over any region O, $F(\underset{\sim}{x})$ can be represented by a second degree polynomial in $\underset{\sim}{x}$, then $V_{\underset{\sim}{x}}$ will also be a second degree polynomial over that region and in addition must be convex (see Appendix 3).

Another source of information about $V_{\underset{\sim}{x}}$ is its behavior in actual examples. This is considered above for the Wheatstone Bridge example. We have also made similar studies for the temperature controller example of Taguchi and Phadke (1984) and for an example, due to Barker (1984). As we have said, for the first example three of the factors behave roughly linearly, and the other two are like a general quadratic. For the latter two examples the functions are even simpler, being monotonic in all the variables.

In spite of the encouraging facts about the objective function $V_{\underset{\sim}{x}}$ we must also accept the fact that situations more difficult than those considered above might occur. Therefore in attempting to find the values $\underset{\sim}{x}$ which minimize $V_{\underset{\sim}{x}}$ we face a familiar dilemma – should we attempt to survey $V_{\underset{\sim}{x}}$ over the *whole design region* or should we rely on the *local properties* of the function? If we do the former then for k not very small it may be very expensive to achieve a grid of points of sufficient density. If we do the latter, then for complex functions, local properties might mislead.

Global Exploration

Taguchi's solution employs global exploration using a three level orthogonal array grid. Since no specific provision is made for interaction between the variables and marginal graphs are used to find the maximum, the method could be expected to work well if, very approximately, the objective function $H_{\underset{\sim}{x}}$ behaved like a sum of independent quadratic components or equivalently like a general second degree expression that does not contain interaction terms – that is to say if

$$H_{\underset{\sim}{x}} \approx \sum_{i=1}^{k}\left(\alpha_i + \beta_i x_i + \beta_{ii} x_i^2\right) = \beta_0 + \sum_{i=1}^{k}\beta_i x_i + \sum_{i=1}^{k}\beta_{ii} x_i^2 \quad (11)$$

However, it can be argued that the need for second order terms of one kind (e.g. quadratic) in an approximating function implies the likely need for second order terms of the other kind (e.g. two-factor interactions); see for example Box and Wilson (1951) and Box and Hunter (1957). More recently one of the present authors wrote "... The way in which variables are chosen is to some extent arbitrary. Thus one investigator might, for example, work with x_1 and x_2, the logs of two concentrations. Another, preferring to think in terms of the ratio and product of these concentrations, might work with $X_1 = x_1 - x_2$ and $X_2 = x_1 + x_2$. The two factor interaction in one parameterization is the difference of the quadratic effects in the other. Thus, particularly when we are dealing with continuous variables and are interested in maxima, it does not make much sense to consider second order terms of one kind without those of the other kind . . ." (Box in discussion of Cox, 1984). Reparameterization in terms of ratios has in fact been used in applications of the parameter design technique; see for example Taguchi and Phadke, (1984). We feel therefore that if second degree power series approximations are to be considered, then a design that allows convenient estimation of two-factor interactions as well as quadratic terms ought to be employed*. One such three-level arrangement for k

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* Barker (1984) in an unpublished manuscript has

= 5 is a composite design consisting of a half fraction of a 2^5 factorial plus 10 axial points constrained to lie in the basic hypercube, plus a center point. This design, like the L_{36} array, uses a particular subset of the full $3^5 = 243$ design points; *but* only 27 runs are required and these are chosen so as to allow all second order terms to be estimated** .

In Appendix 4 we have employed such a design to generate 27 values of H'_x for the Wheatstone Bridge example and have used these points to fit a "response surface" approximation to the actual H'_x surface using ordinary least squares. Figure 7 shows contours of the fitted function in the *CD* plane, with the remaining factors at their optimal levels, for comparison with Figure 4. We see that the optimum point is located quite well, employing a total of only $27 \times 8 = 216$ evaluations of the function $f(x)$.

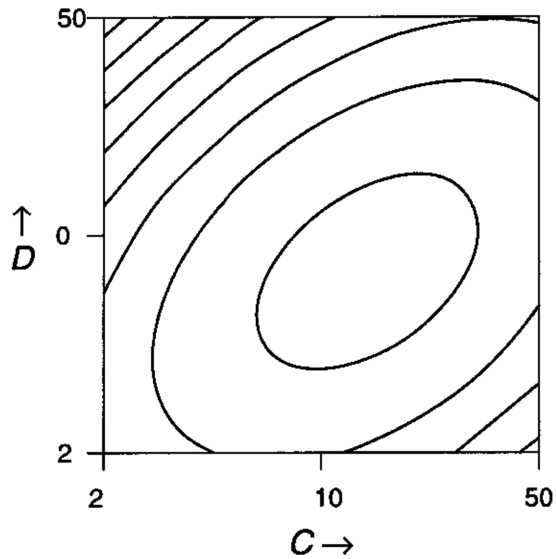

Figure 7. *Contours in the CD plane of the quadratic approximation to H'_x based on a 27-run composite design. A, E, and F are set at their optimal levels.*

RANDOM SEARCH

In most examples some of the "known" quantities (in particular the precise dimensions of the design region and the precise values of $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$ are in reality *not* exactly known. Thus it might be argued that it is pedantic to insist on a precise optimum. A choice of the design variables that is reasonably good might be all that could be expected anyway. In such a case, determination of an objective function H_x at a sample of n points in R and selection of that x yielding the best value of H_x by simple enumeration might be adequate. Sampling could be totally random within R; random on a predecided grid or systematic on a predecided grid such as an orthogonal array. As an example of the last method, in the 36-run inner orthogonal array of Taguchi and Wu, run number 22 gave the highest value of the criterion $10\log_{10}(SN_T)$= 46.5 at the conditions $A_{(-)}$, $C_{(0)}$, $D_{(+)}$, $E_{(+)}$, $F_{(-)}$. This however falls considerably short of the best conditions found by Taguchi's method and even further short of the actual best levels found by direct optimization.

Local Properties

Many optimization strategies are based on local properties of the function. In the purely statistical literature, one method used in both an experimental context (Box and Wilson, 1951) and a computational context (Box and Coutie, 1956) initially employs the method of steepest ascent, switching to the fitting of a second order approximating function when second order terms become important. In the presence of constraints the steepest ascent vector may be projected onto the constraining hyperplanes. Simplex optimization (Spendley, Hext, and Himsworth, 1962; Nelder and Mead, 1964) is another local technique which has been proposed both for experimental and computational purposes. By considering the nature of the function V_x for the Wheatstone Bridge example it is clear that either of these techniques would be effective in quickly finding its minimum. More general local strategies are discussed in the very extensive literature on constrained optimization and nonlinear programming, in particular methods of Fletcher and Powell (1963). See also a comparison of algorithms by M.J. Box (1966) and general discussions by Whittle (1971) and McCormick (1983).

Such methods seem to us much less cumbersome, more computer friendly, and more easily adapted to a range of problems than those employing orthogonal arrays. However, Taguchi and Phadke (1984) criticize these methods as follows: ". . . The advantage of using orthogonal arrays over more commonly used nonlinear programming

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experiment with the use of a 27-run, five-factor composite design as an inner array for the Wheatstone Bridge problem, computing *SN* at each of the 27 points using a 31-point outer array. Although the composite design can of course estimate two-factor interactions, unfortunately he ignores these and fits equation (11) rather than the full quadratic equation. His fitted model consequently is unable to identify the true maximum.

** Another class of three-level layouts that permit the fitting of full second-order equations are the Box-Behnken designs (Box and Behnken, 1960).

methods are:

1) no derivatives have to be computed;
2) hessian does not have to be computed;
3) algorithm is insensitive to starting conditions;
4) large number of variables can be easily handled; and
5) combinations of continuous and discrete variables can be handled easily."

We consider below these various criticisms in the order given.

1) The analytic determination of derivatives would of course be extremely tedious and would not be considered. It is in practice much simpler to determine them numerically, as is for example normally done in nonlinear least squares calculations. When this is done, it is seen that the computation of H or H' from the error transmission formula, and its computation from the summing of squares in (10) in a manner similar to that employed by Taguchi, are essentially identical. However, the use of orthogonal outer arrays for this purpose seems unnecessarily complicated and uneconomical.
2) The evaluation of the hessian determinant of second derivatives would be undertaken to assess the local convexity of the function $V_{\underset{\sim}{x}}$. However, as is shown in Appendix 3, because of the special properties of V_x provided only that the response function $F(\underset{\sim}{x})$ has a local representation in terms of some second degree equation, then V_x must be locally convex and the hessian will always be non-negative.
3) In the general nonlinear programming problem with complicated functions $F(\underset{\sim}{x})$, some assurance can be gained that a genuine minimum has been obtained by initiating the algorithm at different starting points and checking that the same solution is reached. With the functions we have seen for the three examples we have studied, no such difficulty would arise. However, if such assurance was desired the procedure could be run from a number of starting points and because of the very short time required to run the program, the method would still be very fast.
4) It seems to us on the contrary a great deal easier to deal with larger numbers of variables by the method we suggest, which remains essentially the same whatever the number of variables. In particular it is not necessary to seek out special orthogonal arrays appropriate to different examples.
5) When continuous and discrete variables occur as quantitative and qualitative factors, respectively, difficulties can arise because of interaction between the two kinds of factors(the functions involving the quantitative factors are different at different levels of the qualitative factors). When this is so, separate optimizations would have to be conducted at the various levels of the qualitative variables in this case. In view of the great speed of the NLP algorithm, this would normally not be unduly time consuming. However, in these circumstances averaging over orthogonal arrays could clearly yield wrong conclusions.

VARIATIONS FOR FURTHER STUDY

In the above, the response function $y = f(\underset{\sim}{x})$ and the covariance matrix Σ_x are supposed known. This is the situation considered by Taguchi and Wu (1979) in Section 5 of their book, and by Taguchi and Phadke (1984). However, in preliminary studies we have found that parameter design solutions can be dramatically affected by small changes in the assumed magnitudes of the error variances and by changes in how Σ_x depends on $\underset{\sim}{x}$ (Taguchi nearly always assumes proportional errors). We plan to examine this issue of sensitivity to assumptions in a future publication.

Other variants of the problem can also occur. Suppose the function depends on certain constants $\underset{\sim}{\theta}$ so that $y = f(\underset{\sim}{x} \mid \underset{\sim}{\theta})$. Then

1) the *form* of $f(\underset{\sim}{x} \mid \underset{\sim}{\theta})$ may be known, but the constants $\underset{\sim}{\theta}$ may need to be estimated from data. For example, using kinetic theory the yield y of an intermediate product in a consecutive chemical reaction obeying first order kinetics might depend on x_1 = reaction time and x_2 = absolute temperature in the following manner (see Box and Lucas, 1959):

$$y = \frac{k_1(x_2)}{k_1(x_2) - k_2(x_2)}\left(\exp\left(-k_2(x_2)x_1\right) - \exp\left(-k_1(x_2)x_1\right)\right) \quad (12)$$

where

$$k_1(x_2) = \alpha_1 \exp\left(\frac{-\beta_1}{x_2}\right)$$

$$k_2(x_2) = \alpha_2 \exp\left(\frac{-\beta_2}{x_2}\right)$$

and where α_1, α_2, β_1, β_2 are constants to be estimated from suitable data;

2) the nature of the function $f(\underset{\sim}{x}|\underset{\sim}{\theta})$ might be unknown, but might be approximated by some graduating function $g(\underset{\sim}{x}|\underset{\sim}{\beta})$ such as a polynomial. Since the possibility of optimization depends on the nonlinearity of the function; f would need to be nonlinear in $\underset{\sim}{x}$ and if g were a polynomial it would need to be of at least second degree; or

3) in either of the above situations the covariance matrix $\Sigma_{\underset{\sim}{x}}$ might be known or it might be unknown, so that it too must be estimated from the data.

It is planned to consider these problems in later publications.

CONCLUSION

We feel that the parameter design problem enunciated by Taguchi is of great practical importance; however, we believe that it is better tackled in the manner outlined above employing standard nonlinear programming (constrained optimization) methods.

ACKNOWLEDGMENTS

This research was sponsored by the United States Army under contract number DAAG29-80-C-0041, by the National Science Foundation under grant number DMS-8420968, and by the Vilas Trust of the University of Wisconsin-Madison.

APPENDIX 1: THE TRANSMITTED VARIATION FUNCTION FOR THE WHEATSTONE BRIDGE EXAMPLE

Here we analyze the transmitted variation function for the Wheatstone Bridge example, considering in detail how errors enter in the determination of y. We will deal with the factors A, B, C, D, E, F, X in the original units used by Taguchi and Wu in Tables 1(a) and 1(b).

Consider Equation (2) that, given a true resistance p_0, becomes an identity that relates together the true but unknown values of all the components of the network. Writing $\alpha, \beta, \gamma, \delta, \varepsilon, \phi, \chi$ for the true values of the nominal factor levels A, B, C, D, E, F, X, and substituting into (2)

$$p_0 = \frac{\beta\delta}{\gamma} - \frac{\chi}{\gamma^2\varepsilon}\left[\alpha(\gamma+\delta)+\delta(\beta+\gamma)\right] \cdot \left[\beta(\gamma+\delta)+\phi(\beta+\gamma)\right] \quad \text{(A1.1)}$$

Equivalently

$$\begin{aligned} 0 &= h(\alpha, \beta, \gamma, \delta, \varepsilon, \phi, \chi) \\ &= \frac{\beta\delta}{\gamma} - \frac{\chi}{\gamma^2\varepsilon}\left[\alpha(\gamma+\delta)+\delta(\beta+\gamma)\right] \quad \text{(A1.2)} \\ &\quad \cdot \left[\beta(\gamma+\delta)+\phi(\beta+\gamma)\right] - p_0. \end{aligned}$$

Thus each factor is defined implicitly by the remaining factors. In particular, any deviation in $\alpha, \gamma, \delta, \varepsilon, \phi$, or χ will cause a compensatory deviation in β.

Now y is an estimate of p_0 calculated from Equation (1) (which is Equation A1.2 evaluated at the nominal factor levels $\underset{\sim}{z} = (A, B, C, D, E, F; X = 0)'$):

$$y = \frac{BD}{C} = (\beta_0 + e_B)\frac{D}{C}. \quad \text{(A1.3)}$$

The term e_B denotes the overall error in B and consists of a random component e_B which may arise from misreading or miscalibration of the potentiometer dial plus a sum of small biases transmitted to β via the equation (A1.2) from errors in all the other factors. Given the small error variations in the factors in this example we have, to a good approximation,

$$e_B = \varepsilon_B + \left(\left.\frac{\partial\beta}{\partial\alpha}\right|_{\underset{\sim}{z}} \varepsilon_A + \left.\frac{\partial\beta}{\partial\gamma}\right|_{\underset{\sim}{z}} \varepsilon_C + \left.\frac{\partial\beta}{\partial\delta}\right|_{\underset{\sim}{z}} \varepsilon_D + \left.\frac{\partial\beta}{\partial\varepsilon}\right|_{\underset{\sim}{z}} \varepsilon_E + \left.\frac{\partial\beta}{\partial\phi}\right|_{\underset{\sim}{z}} \varepsilon_F + \left.\frac{\partial\beta}{\partial\chi}\right|_{\underset{\sim}{z}} \varepsilon_X \right) \quad \text{(A1.4)}$$

where the partial derivatives are of the implicit function β, evaluated at $\underset{\sim}{z}$. It can be shown that the conditions on h required for implicit differentiation of β are satisfied so that

$$\left.\frac{\partial\beta}{\partial\alpha}\right|_{\underset{\sim}{z}} = \frac{g_A(\underset{\sim}{z})}{g_B(\underset{\sim}{z})}, \left.\frac{\partial\beta}{\partial\gamma}\right|_{\underset{\sim}{z}} = -\frac{g_C(\underset{\sim}{z})}{g_B(\underset{\sim}{z})}, \text{ etc.,} \quad \text{(A1.5)}$$

where

$$g_A(\underset{\sim}{z}) = \left.\frac{\partial h}{\partial\alpha}\right|_{\underset{\sim}{z}}, g_B(\underset{\sim}{z}) = \left.\frac{\partial h}{\partial\beta}\right|_{\underset{\sim}{z}}, \text{ etc.} \quad \text{(A1.6)}$$

Thus

$$\begin{aligned} e_B &= \varepsilon_B - \frac{1}{g_B(\underset{\sim}{z})}\left[g_A(\underset{\sim}{z})\varepsilon_A + g_C(\underset{\sim}{z})\varepsilon_C + \ldots + g_X(\underset{\sim}{z})\varepsilon_X\right] \\ &= \frac{-1}{g_B(\underset{\sim}{z})}\left[g_A(\underset{\sim}{z})\varepsilon_A - g_B(\underset{\sim}{z})\varepsilon_B + g_C(\underset{\sim}{z})\varepsilon_C + \ldots + g_X(\underset{\sim}{z})\varepsilon_X\right] \end{aligned} \quad \text{(A1.7)}$$

whence

$$\text{var}(e_B) = \frac{1}{g_B^2(\underset{\sim}{z})}\left[g_A^2(\underset{\sim}{z})\sigma_A^2 + g_B^2(\underset{\sim}{z})\sigma_B^2 + g_C^2(\underset{\sim}{z})\sigma_C^2 + \ldots + g_X^2(\underset{\sim}{z})\sigma_X^2\right]. \quad \text{(A1.8)}$$

Finally we can write the criterion

$$V_{\underset{\sim}{z}} = \text{var}\left(\ln y_{\underset{\sim}{z}}\right) = \text{var}\left[\ln(B) + \ln\left(\frac{D}{C}\right)\right]$$
$$= \frac{1}{B^2}\text{var}(B) = \frac{1}{B^2}\text{var}(e_B)$$
$$= \frac{1}{B^2 g_B^2\left(\underset{\sim}{z}\right)}\left[g_A^2\left(\underset{\sim}{z}\right)\sigma_A^2 + g_B^2\left(\underset{\sim}{z}\right)\sigma_B^2 + g_C^2\left(\underset{\sim}{z}\right)\sigma_C^2 + \ldots + d_X^2\left(\underset{\sim}{z}\right)\sigma_X^2\right] \quad \text{.(A1.9)}$$

Now in the present example the target resistance for which maximum precision is to be achieved is $p_0 = 2$ ohms. It follows from (A1.2) that, approximately, $B = 2C/D$ and that $g_B(\underset{\sim}{z}) = D/C$ is the nominal value of X is 0. Substituting these expressions into (A1.9) yields our criterion in operational form,

$$V_{\underset{\sim}{z}} = \frac{1}{4}\left[g_A^2\left(\underset{\sim}{z}\right)\sigma_A^2 + g_B^2\left(\underset{\sim}{z}\right)\sigma_B^2 + g_C^2\left(\underset{\sim}{z}\right)\sigma_C^2 + \ldots + g_X^2\left(\underset{\sim}{z}\right)\sigma_X^2\right] \quad \text{(A1.10)}$$

which is seen to be the square of the coefficient of variation of y where now the separate contributions of all the error factors to variation in y are clearly shown.

Alternatively V_x can be represented directly in terms of the differential transmission of error to ln$y$ as in (6) by writing

$$V_{\underset{\sim}{z}} = \text{var}\left(\ln y_{\underset{\sim}{z}}\right)$$
$$= d_A^2\left(\underset{\sim}{z}\right)\sigma_A^2 + d_B^2\left(\underset{\sim}{z}\right)\sigma_B^2 + \ldots + d_X^2\left(\underset{\sim}{z}\right)\sigma_X^2 \quad \text{(A1.11)}$$

where now

$$d_i\left(\underset{\sim}{z}\right) = \left.\frac{\partial \ln h}{\partial z_i}\right|_{\underset{\sim}{z}} \quad i = 1, \ldots, 7. \quad \text{(A1.12)}$$

APPENDIX 2: COMPARISON OF V_x BASED ON NUMERICAL DERIVATIVES VERSUS TAGUCHI'S SN_T BASED ON OUTER ARRAYS

We used equation (10) with $\alpha = 10^{-3}$ to compute $H_x' = 10\log_{10} V_x$ for comparison with the results obtained by Taguchi and Wu at their L_{36} inner array points (their values were reported to one decimal place).

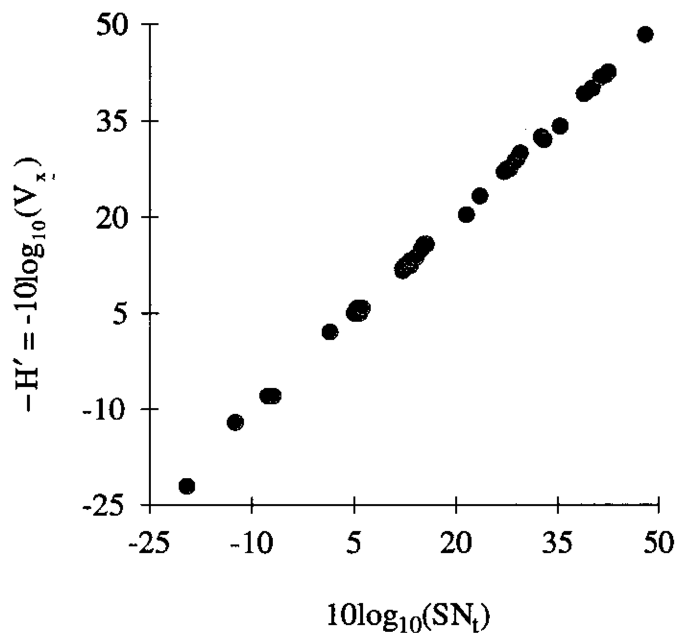

Figure A2.1 *Minitab plot of* $-10\log_{10} V_x$ *versus* $10\log_{10}(SN_T)$.

The approximation is very good, as shown in Figure A2.1. The correlation between the two sets of numbers is greater than .9999.

Table A2.1

Comparison of $-H_x = 10\log_{10} V_x$ *obtained from divided first differences of* $Y = \ln y$, *against Taguchi's criterion* $10\log_{10}(SN_T)$ *obtained from outer arrays.*

RUN NO.	*FROM FIRST DIFFS*	*FROM OUTER ARRAYS*
1	32.27	32.2
2	26.68	26.7
3	15.90	15.9
4	36.43	36.4
5	28.59	28.6
6	7.23	7.2
7	16.48	16.5
8	13.02	13.0
9	28.05	28.0
10	15.07	15.0
11	16.47	16.4
12	25.55	25.5
13	43.75	43.8
14	−8.28	−8.3
15	14.60	14.6
16	28.98	29.0
17	6.93	6.9
18	14.70	14.7
19	21.48	21.5
20	17.39	17.4
21	13.98	14.0
22	46.39	46.5
23	5.53	5.5
24	−8.15	−8.2
25	27.35	27.3
26	43.50	43.4
27	−20.85	−20.9
28	44.09	44.1
29	39.33	39.3
30	−17.00	−17.0
31	23.07	23.0
32	44.19	44.2
33	−0.84	−0.9
34	43.50	43.4
35	−7.64	−7.7
36	7.97	8.0

APPENDIX 3: LINKS BETWEEN THE TRANSMITTED VARIATION FUNCTION AND THE RESPONSE FUNCTION

Here we discuss briefly how the transmitted variation function $V_x = \text{var}\left(Y_x\right)$ is linked to the response function $Y = F(x)$. Some of the helpful properties of V_x that arise when $F(x)$ has a quadratic approximation are also described.

The following notation will hold throughout:

- b's will denote *coefficients* of the *response function* $F(x)$ (when a polynomial representation is available);
- d's will denote *derivatives* of the *response function*;
- c's will denote *coefficients* of the *transmitted variation function*; and
- g's will denote *gradients* of the *transmitted variation function*.

The usual error propogation formula gives, locally and approximately,

$$\begin{aligned} V_x &= d_1^2\left(x\right)\sigma_1^2 + d_2^2\left(x\right)\sigma_2^2 + \ldots + d_k^2\left(x\right)\sigma_k^2 \\ &= d_x' \Sigma d_x \end{aligned} \qquad \text{(A3.1)}$$

where k is the number of factors x that transmit error to the response y, d_x is the vector $(d_1(x), \ldots, d_k(x))'$ of partial derivatives

$$d_i\left(x\right) = \left.\frac{\partial F\left(x\right)}{\partial x_i}\right|_x \quad i = 1, 2, \ldots, k, \qquad \text{(A3.2)}$$

and Σ is the $k \times k$ diagonal covariance matrix for the errors in x, whose i^{th} element is $\sigma_i^2 > 0$. We assume that the errors in the x's are independent and that the x's have been transformed so that the σ_i^2's do not depend on x.

MINIMIZATION OF V_x USING LOCAL PROPERTIES

Suppose the object is to minimize the transmitted variation function V_x by a method such as steepest decent. To this end the gradients of V_x are required. For the case $k = 2$, we have

$$\begin{aligned} g_1\left(x\right) &= \left.\frac{\partial V_x}{\partial x_1}\right|_x = \left.\frac{\partial}{\partial x_1}\left(d_1^2\left(x\right)\sigma_1^2 + d_2^2\left(x\right)\sigma_2^2\right)\right|_x \\ &= 2d_1\left(x\right)d_{11}\left(x\right)\sigma_1^2 + 2d_2\left(x\right)d_{12}\left(x\right)\sigma_2^2 \\ g_2\left(x\right) &= \left.\frac{\partial V_x}{\partial x_2}\right|_x = \left.\frac{\partial}{\partial x_2}\left(d_1^2\left(x\right)\sigma_1^2 + d_2^2\left(x\right)\sigma_2^2\right)\right|_x \\ &= 2d_1\left(x\right)d_{12}\left(x\right)\sigma_1^2 + 2d_2\left(x\right)d_{22}\left(x\right)\sigma_2^2 \end{aligned} \qquad \text{(A3.3)}$$

where $d_{ij}\left(x\right)$ are the second partial derivatives of $F(x)$ evaluated at x. More generally if g_x is the vector of gradients $(g_1(x), \ldots, g_k(x))'$ of V_x, we have

$$g_x = 2D_x \Sigma d_x \qquad \text{(A3.4)}$$

where D_x is a matrix with ij^{th} element $d_{ij}(x)$.

Thus local evaluation of g_x requires only the first two derivatives of the response function $F(x)$. In practice the derivatives need only be approximated numerically and to do so requires no more than a layout of points sufficient to fit a second-order approximation of $F(x)$.

IMPLICATIONS OF A QUADRATIC $F(x)$

We now consider the special case where the response function $F(x)$ is exactly quadratic. Write

$$F\left(x\right) = b_0 + x'b + x'Bx \qquad \text{(A3.5)}$$

where $b = (b_1, b_2, \ldots, b_k)'$ and

$$B = \begin{pmatrix} b_{11} & \frac{1}{2}b_{12} & \cdots & \frac{1}{2}b_{1k} \\ \frac{1}{2}b_{21} & b_{22} & \cdots & \frac{1}{2}b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}b_{k1} & \frac{1}{2}b_{k2} & \cdots & b_{kk} \end{pmatrix}.$$

Then the vector d_x of first derivatives of $F(x)$ is

$$d_x = b + 2B_x \qquad \text{(A3.6)}$$

whence

$$\begin{aligned} V_x &= d_x'\Sigma d_x \\ &= \left(b + 2B_x\right)' \Sigma \left(b + 2B_x\right) \\ &= b'\Sigma b + x'(4B\Sigma B) + x'(4B\Sigma B)x \\ &= c_0 + x'c + x'Cx \end{aligned} \qquad \text{(A3.7)}$$

where $c_0 = b'\Sigma b$, $c = 4B\Sigma b$, and $C = 4B\Sigma B$.

Thus when the response function $F(x)$ is quadratic, the transmitted variation function V_x is quadratic as well, and the coefficients that are simple functions of the b's and Σ. Moreover V_x is always convex since its hessian matrix of second partial derivatives $2C = 8B\Sigma B$ is always positive semi-definite. The determinant of the hessian is positive unless $\det(B) = 0$.

A zero value for the hessian determinant is not informative about the existence of a unique minimum of V_x in the region R. For example, a zero determinant results when V_x is a stationary ridge so that a multiplicity of "best" factor settings is available; on the other hand, the determinant also vanishes in cases when a unique minima in R do exist (for example when V_x is a nonstationary ridge or when one or more x's has no second-order effect on V_x). Thus the routine calculation of the hessian determinant as a diagnostic would in any case be of limited use.

After final convergence of any procedure, numerical determination of second derivatives (using for example a Koshal design (Koshal, 1933; Kanemasu, 1979)) and canonical analysis should be routinely carried out to elucidate the nature of the stationary regions (Box and Wilson, 1951; Box and Draper, 1986). The discovery of a stationary ridge system pointing to a line, plane, or hyperplane of solutions, is a bonus showing the alternative solutions available. Furthermore, such an analysis could discover the existence of a rising ridge system pointing to the desirability, if possible, of extending the design region R in specific directions.

APPENDIX 4: A "RESPONSE SURFACE" APPROXIMATION TO THE OBJECTIVE FUNCTION SURFACE

As previously discussed the design factor settings that maximize the objective function $10\log_{10}(SN_T)$ can be located quite well by global exploration of the design region R, by fitting a full second-degree equation to a set of strategically placed points. We have done this with the 27-run 3-level composite design shown in Table A4.1. We have used the objective function $-H'_x$ that closely approximates $10\log_{10}(SN_T)$ (see Appendix 2), but that requires only an 8-point layout. Thus a total of $8 \times 27 = 216$ evaluations of y were required, in contrast with the 1296 runs required by the approach of Taguchi and Wu.

Table A4.1.

A 27-run 3-level composite design.

3-LEVEL COMPOSITE DESIGN						
Run No.	*A*	*C*	*D*	*E*	*F*	*–H′*
1	–1	–1	–1	–1	1	10.1348
2	1	–1	–1	–1	–1	5.2513
3	–1	1	–1	–1	–1	20.4796
4	1	1	–1	–1	1	–14.6555
5	–1	–1	1	–1	–1	15.8113
6	1	–1	1	–1	1	–33.4252
7	–1	1	1	–1	1	9.1047
8	1	1	1	–1	–1	7.2276
9	–1	–1	–1	1	–1	47.1396
10	1	–1	–1	1	1	10.9286
11	–1	1	–1	1	1	39.0830
12	1	1	–1	1	–1	21.9384
13	–1	–1	1	1	1	21.4846
14	1	–1	1	1	–1	16.5998
15	–1	1	1	1	–1	46.8175
16	1	1	1	1	1	15.8954
17	–1	0	0	0	0	38.3359
18	1	0	0	0	0	13.1376
19	0	–1	0	0	0	18.4165
20	0	1	0	0	0	27.2549
21	0	0	–1	0	0	25.3242
22	0	0	1	0	0	18.8845
23	0	0	0	–1	0	12.7334
24	0	0	0	1	0	39.8871
25	0	0	0	0	–1	34.4269
26	0	0	0	0	1	14.6698
27	0	0	0	0	0	26.6807

Writing $x_1, x_2, \ldots, x_5$ to represent the five design factors *A, C, D, E,* and *F* logged, scaled, and centered to cover the range –1 to +1, the fitted equation

$$-\hat{H}'_x = c_0 + \sum_{i=1}^{5} c_i x_i + \sum_{i \le j} c_{ij} x_i x_j \qquad \text{(A4.1)}$$

was obtained by ordinary least squares. The estimated coefficients are displayed in Table A4.2.

Table A4.2

Coefficients of the fitted function (A4.1)

COLUMN	*COEFFICIENT*
	26.6428
*C*1	–11.4163
*C*2	3.3780
*C*3	–2.6235
*C*4	12.6173
*C*5	–7.9151
*C*11	–0.901
*C*22	–3.802
*C*33	–4.534
*C*44	–0.328
*C*55	–2.090
*C*12	0.6336
*C*13	0.4034
*C*14	0.1231
*C*15	–1.3646
*C*23	4.0740
*C*24	0.1998
*C*25	1.7902
*C*34	0.2625
*C*35	–1.5049
*C*45	2.0317

The major features of the objective function surface can easily be seen: x_1, x_4, and x_5 (*A, E, F*) are essentially linear; and x_2 and x_3 (*C* and *D*) have a quadratic tendency and a large interaction between them.

Thus the "optimal" levels of the design factors can be approximated by maximizing the fitted equation in *C* and *D* while holding *A, E,* and *F* at their best levels on the boundaries of R: $A_{(-)}$, $E_{(+)}$, $F_{(-)}$. The contours of the fitted surface in the *CD* plane are shown in Figure 7.

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